

SOME EXAMPLES OF LOCALLY DIVIDED RINGS

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To Jim Huckaba, on the occasion of his retirement

ABSTRACT: New classes of locally divided rings R are introduced in two ways: by requiring all *CPI*-extensions of R to be special types of rings of fractions and by requiring all localizations of R to be pseudo-valuation rings. For rings R whose zero-divisors are nilpotent and whose minimal prime ideal is divided, the first method characterizes the locally divided R in which each nonminimal prime ideal is contained in a unique maximal ideal. The second method, concerning *LPVRs*, focuses on idealization, for which a typical result is the following. If E is a module over an integral domain R , then $R(+)E$ is an *LPVR* if and only if R is an *LPVD* and E_M is a divisible R_M -module for each maximal ideal M of R .

1 INTRODUCTION

All rings considered below are commutative with identity, typically nonzero. If A is a ring, then $Z(A)$ denotes the set of zero-divisors of A , $Nil(A)$ the set of nilpotent elements of A , $Spec(A)$ the set of prime ideals of A , $Max(A)$ the set of maximal ideals of A , $tq(A)$ the total quotient ring of A , and $dim(A)$ the (Krull) dimension of A . Our main purpose here is to further the studies in [3] by identifying some new classes of locally divided rings. This is done in Section 2 by focusing on rings of fractions, with characterizations of some classes of locally divided rings in Theorem 2.2 (b) and Theorem 2.4 (b); and in Section 3 by introducing the class of locally pseudo-valuation rings (*LPVRs*), with characterizations of the *LPVRs* among certain classes of idealizations in Proposition 3.4 (f), (g).

We next recall some background on divided and locally divided rings. These concepts were introduced for integral domains in [7], and their final extension for rings was achieved in [3]. (For additional background, see the summary in [6, Section 7].) Let A be a ring. We say that $P \in Spec(A)$ is a *divided prime (ideal of A)* if P is comparable under inclusion with each (equivalently, each principal) ideal of A ; that A is a *divided ring* if each $P \in Spec(A)$ is divided in A ; and that A is a

locally divided ring if A_P is a divided ring for each $P \in \text{Spec}(A)$. Of course, each divided ring is quasilocal. Moreover, each locally divided ring is a treed ring (in the sense that none of its maximal ideals can contain incomparable prime ideals) and also a going-down ring (in the sense of [11]) [3, Proposition 3.1]. In contrast to the situation for integral domains, neither one-dimensional rings nor Prüfer rings need be locally divided [3, Example 2.18]. However, domain-like behavior often occurs in rings A such that $Z(A) = \text{Nil}(A)$: cf. [11, Corollary 2.6], [3, Proposition 2.19 (b)], [3, Theorem 2.10]. Rings A such that $Z(A) = \text{Nil}(A)$ are the (nonzero) rings in which 0 is a primary ideal. We often use the facts that if A is a ring such that $Z(A) = \text{Nil}(A)$, then A has a unique minimal prime P_0 ; $tq(A)$ can be identified as A_{P_0} ; and, more generally, for each $P \in \text{Spec}(A)$, the localization A_P can be viewed, up to canonical A -algebra isomorphism, as an overring of A (cf. [3, Proposition 2.5]).

Section 2 focuses exclusively on rings R such that $Z(R) = \text{Nil}(R)$. For such rings, it was shown in [3, Theorem 2.10] that R is a locally divided ring if and only if each *CPI*-extension of R (in the sense of [4]) is a ring of fractions R_S of R ; this generalized a result on integral domains [9, Theorem 2.4]. Theorems 2.2 (b) and 2.4(b) identify the classes of locally divided rings R for which such S can be chosen as either the complement of a prime ideal or the set of integral powers of an element of R . Along the way, contact is made in Theorem 2.5 with topological ideas involving the so-called *g*-rings.

An important class of divided integral domains is provided by the pseudo-valuation domains (*PVDs*) introduced by Hedstrom-Houston. The pseudo-valuation concept was extended from integral domains to the context of rings with zero-divisors in [1], through the introduction of pseudo-valuation rings (*PVRs*), a class of divided rings whose definition is recalled in Section 3. That section's main purpose is to globalize the "*PVR*" concept, thereby obtaining a new class of locally divided rings. As motivation, note that an important globalization of the *PVD* concept was provided by the locally pseudo-valuation domains (*LPVDs*) introduced in [12], [13]. Accordingly, it seems natural to make the following definition: a ring R is said to be a *locally pseudo-valuation ring (LPVR)* if R_P is a *PVR* for each $P \in \text{Spec}(R)$. Evidently, each *LPVR* is locally divided, but the converse is false, even in the case of quasilocal integral domains [8, Remark 4.10 (b)]. A direction for study is suggested by the role of idealizations in developing characterizations of locally divided rings and identifying new classes of locally divided rings [3, Proposition 2.16]. For this reason, we develop the above-mentioned characterizations of *LPVR* idealizations in Proposition 3.4 (f), (g). For additional examples of non-domain *LPVRs*, we draw attention to Corollary 3.3, Example 3.5 (b) and Proposition 3.6.

Any unexplained material is as in [14], [16], [17].

2 WHEN *CPI*-OVERRINGS ARE SPECIAL RINGS OF FRACTIONS

We begin by generalizing a result that was originally proved for integral domains [8, Proposition 2.1]. Theorem 2.1 may be viewed as a companion for [3, Theorem 3.3].

THEOREM 2.1. *Let R be a ring such that $Z(R) = Nil(R)$. Then $R + PR_P$ is integral over R for each $P \in Spec(R)$ if and only if R is a quasilocal going-down ring.*

Proof. Suppose first that each CPI -overring $R + PR_P$ is integral over R . We claim that R is quasilocal. If not, choose distinct $M, N \in Max(R)$. Pick $u \in M \setminus N$ and $v \in N \setminus M$. By hypothesis, $w := uv^{-1} \in MR_M$ is integral over R . Fix an integrality equation, $w^n + r_{n-1}w^{n-1} + \dots + r_1w + r_0 = 0$, with $r_i \in R$ for each i . Multiplying by v^n and viewing matters via the canonical inclusions $R \subseteq R_M \subseteq tq(R)$, we have that $u^n \in Rv \subseteq N$, whence $u \in N$, the desired contradiction, thus proving the above claim. According to [11, Proposition 2.1 (a)], it remains only to prove that if P_0 denotes the unique minimal prime ideal of R , then $D := R/P_0$ is a going-down domain. Since D inherits the quasilocal property from R , [8, Proposition 2.1] reduces our task to showing that if $P \in Spec(R)$, then $Q := P/P_0$ is such that $D + QD_Q$ is integral over D . Since each element of PR_P is assumed integral over R , the assertion then follows via the canonical identification $QD_Q \cong Q(R_P/P_0R_P) = PR_P/P_0R_P$. (cf. [5, Proposition 11 (i), p. 70]).

For the converse, we suppose that R is a quasilocal going-down ring and proceed to show that if $P \in Spec(R)$, then each element $w \in PR_P$ is integral over R . Let P_0, D, Q be as above. View $\bar{w} := w + P_0R_P \in PR_P/P_0R_P \cong QD_Q$. Since D is a quasilocal going-down domain, [8, Proposition 2.1] produces $n \geq 1$ such that $\bar{w}^n \in Q = P/P_0 \subseteq D \cong (R + P_0R_P)/P_0R_P$. Thus, there exists $p \in P$ such that $w^n - p \in P_0R_P = Nil(R_P)$. Therefore, $(w^n - p)^m = 0$ for some $m \geq 1$, whence w is integral over R . \square

A proper overring cannot be both integral and flat [15, Proposition 12]. Accordingly, one might guess that flatness of CPI -overrings would characterize rings that are rather different from those discussed in Theorem 2.1. In fact, [3, Theorem 2.10] establishes that a special family of going-down rings, namely, the locally divided rings, is characterized, among rings R such that $Z(R) = Nil(R)$, by the requirement of having flat CPI -overrings; and, moreover, that in this case, the flat structures in question arise as rings of fractions. In Theorems 2.2 and 2.4, we sharpen the focus by requiring particular types of rings of fractions.

THEOREM 2.2. *Let R be a ring such that $Z(R) = Nil(R)$. Then:*

(a) *If R is a locally divided ring and $P \in Spec(R)$, then $R + PR_P = R_S$, where the multiplicatively closed set S is defined by $S = R \setminus \cup\{M \mid P \subseteq M, M \in Max(R)\}$. (b) Let P_0 denote the unique minimal prime ideal of R . Suppose that $R + P_0R_{P_0}$ is R -flat (this holds if, for instance, P_0 is a divided prime ideal of R). Then the following conditions are equivalent:*

- (1) *For each $P \in Spec(R) \setminus \{P_0\}$, there exists $Q \in Spec(R)$ such that $R + PR_P = R_Q$;*
- (2) *For each $P \in Spec(R) \setminus \{P_0\}$, there exists a unique $M \in Max(R)$ such that $R + PR_P = R_M$;*
- (3) *R is a locally divided ring, and for each $P \in Spec(R) \setminus \{P_0\}$, there exists a unique $M \in Max(R)$ such that $P \subseteq M$.*

Proof. (a) The argument given in [9, Proposition 2.3] carries over from the context of integral domains, *mutatis mutandis*.

(b) We first dispatch the parenthetical assertion: $P_0R_{P_0} = P_0$ if (and only if) P_0

is a divided prime ideal of R [3, Proposition 2.5 (c)], in which case $R + P_0R_{P_0} = R$ is certainly R -flat. Next, observe that if P_1 and P_2 are prime ideals of R such that $R_{P_1} = R_{P_2}$, then $P_1 = P_1R_{P_1} \cap R = P_2R_{P_2} \cap R = P_2$. Consequently, (3) \Rightarrow (2) by (a). Moreover, it is trivial that (2) \Rightarrow (1). Finally, assume (1). By [3, Theorem 2.10], R is a locally divided ring. Fix $P \in \text{Spec}(R)$, $P \neq P_0$. By (a) and (1), there exists $Q \in \text{Spec}(R)$ such that $R_Q = R + PR_P = R_S$, where $S = R \setminus \cup\{M \mid P \subseteq M, M \in \text{Max}(R)\}$. As $R_Q \subseteq R_P$, we have $PR_P \cap R_Q \subseteq QR_Q$, and so intersecting with R yields that $P \subseteq Q$. Next, consider any $N \in \text{Max}(R)$ such that $Q \subseteq N$. By [3, Proposition 2.5 (b)], $R_N \subseteq R_Q = R_S$. However, it follows from the definition of S that $R_S \subseteq R_N$, and so $R_N = R_Q$. Then, by the above observation, we see that $N = Q$; in particular, $Q \in \text{Max}(R)$. It remains only to show that if $M \in \text{Max}(R)$ and $P \subseteq M$, then $M = Q$. Since $R_Q = R_S \subseteq R_M$, we see, by reasoning as above, that $M \subseteq Q$. Thus, we are done by the maximality of M . \square

We pause to make two comments about the formulation of Theorem 2.2. First, it follows from (a) and [3, Proposition 2.5 (c)] that if the equivalent conditions in (b) are satisfied (and R is as assumed), then P_0 is a divided prime ideal of R . Second, the stipulation " $P \neq P_0$ " appears in (b) in order to avoid reducing to the quasilocal case. Indeed, if the condition in (2) were asserted for $P = P_0$, it would follow (for instance, via pullback-theoretic reasoning as in [4, Theorem 2.5]) that M is the only maximal ideal of R .

COROLLARY 2.3. *Let R be a ring such that $Z(R) = \text{Nil}(R)$. Then $R + PR_P = R_P$ for each $P \in \text{Spec}(R)$ if and only if $\dim(R) = 0$.*

Proof. If $\dim(R) = 0$, then $\text{Spec}(R)$ consists of just P_0 , the unique minimal prime ideal of R , in which case $R_{P_0} = R = R + P_0 = R + P_0R_{P_0}$. Conversely, suppose that $R + PR_P = R_P$ for each $P \in \text{Spec}(R)$. Fix $Q \in \text{Spec}(R)$. It suffices to show that $Q \in \text{Max}(R)$. We do so by revisiting some of the ideas in the proof of Theorem 2.2 (b). Choose $N \in \text{Max}(R)$ such that $Q \subseteq N$. It is enough to show that $R_Q = R_N$ (for then $Q = N$ is a maximal ideal of R). This, in turn, follows since $R_Q = R_{R \setminus \cup\{M \mid Q \subseteq M, M \in \text{Spec}(R)\}} \subseteq R_N \subseteq R_Q$: the equality holding by Theorem 2.2 (a), the first inclusion by the definition of localization, and the second inclusion by [3, Proposition 2.5 (a)]. \square

It may be of interest to record the following generalization of Corollary 2.3. Let R be a ring such that $Z(R) = \text{Nil}(R)$ and let $P \in \text{Spec}(R)$. Then $R + PR_P = R_P$ if and only if $P \in \text{Max}(R)$. For a proof, note that one can adapt the argument, *mutatis mutandis*, that was given for integral domains in [10, Proposition 2.8 (a)].

THEOREM 2.4. *Let R be a quasilocal ring such that $Z(R) = \text{Nil}(R)$. Then:*

(a) *Let $P \in \text{Spec}(R)$. Then there exists a non-zero-divisor $r \in R$ such that $R + PR_P = R_r$ if and only if P is a divided prime ideal of R .*

(b) *For each $P \in \text{Spec}(R)$, there exists a non-zero-divisor $r \in R$ such that $R + PR_P = R_r$ if and only if R is a divided ring (if and only if R is a locally divided ring).*

Proof. (a) If P is a divided prime ideal of R , then $PR_P = P$ [3, Proposition 2.5 (c)], whence $R + PR_P = R = R_1$. Conversely, suppose that $R + PR_P = R_r$ for some non-zero-divisor $r \in R$. We show that P is divided in R ; equivalently, by [3,

Proposition 2.5 (c)], that $PR_P \subseteq R$. It suffices to establish that r is a unit of R . Deny. Since r is a non-zero-divisor, we have $r^{-1} \in R + PR_P$; that is, there exist $d \in R, p \in P$ and $z \in R \setminus P$ such that $r^{-1} = d + pz^{-1}$ in $tq(R)$. It follows that $z(1 - dr) = rp$. As R is quasilocal and r is a nonunit of R , we have that $1 - dr$ is a unit of R , and so $z = (1 - dr)^{-1}rp \in Rp \subseteq P$, the desired contradiction.

(b) The first equivalence follows from (a), since a ring R is divided if and only if each prime ideal of R is divided in R . The parenthetical equivalence follows since a quasilocal ring is divided if and only if it is locally divided (cf. [11, Remark (c), p. 47]). \square

Considerations involving rings of fractions of the kind figuring in Theorem 2.4 may remind one of g -rings. Recall that a ring R is called a g -ring if, for each $P \in \text{Spec}(R)$, there exists $r \in R \setminus P$ such that $R_P \cong R_r$. Any open domain, in the sense of [19], is a g -ring. The next result pursues the g -ring connection. First, we recall some terminology and notation. Distinct prime ideals $P \subset Q$ of a ring R are said to be *adjacent (in R)* in case no prime ideal of R is properly contained between P and Q . Also, if R is a ring and $r \in R$, then as in [5], X_r denotes the basic Zariski-open set $\{Q \in \text{Spec}(R) \mid r \in R \setminus Q\}$.

THEOREM 2.5. *Let R be a quasilocal treed ring such that $Z(R) = \text{Nil}(R)$. Then the following conditions are equivalent:*

- (1) *R is a g -ring, that is, for each $P \in \text{Spec}(R)$, there exists (a non-zero-divisor) $r \in R \setminus P$ such that $R_P = R_r$;*
- (2) *For each $P \in \text{Spec}(R)$, the set $\{Q \in \text{Spec}(R) \mid Q \subseteq P\}$ is Zariski-open in $\text{Spec}(R)$;*
- (3) *For each $P \in \text{Spec}(R)$, there exists $r \in R \setminus P$ such that $\{Q \in \text{Spec}(R) \mid Q \subseteq P\} = \{Q \in \text{Spec}(R) \mid r \in R \setminus Q\}$;*
- (4) *For each $P \in \text{Spec}(R)$, one has $P \subset \cap\{Q \in \text{Spec}(R) \mid P \subset Q\}$;*
- (5) *For each $P \in \text{Spec}(R) \setminus \text{Max}(R)$, there exists $Q \in \text{Spec}(R)$ such that $P \subset Q$ are adjacent in $\text{Spec}(R)$.*

Proof. We first comment on the parenthetical phrase in (1). If $P \in \text{Spec}(R)$, $r \in R$ satisfy $R_P = R_r$, then r is a unit of R_P , whence $r \in R \setminus P \subseteq R \setminus \text{Nil}(R) = R \setminus Z(R)$; that is, r is a non-zero-divisor in R .

(1) \Rightarrow (2): If P, r are as in (1), then $\{Q \in \text{Spec}(R) \mid Q \subseteq P\} = X_r$, a Zariski-open subset of $\text{Spec}(R)$.

(2) \Rightarrow (3): Fix $P \in \text{Spec}(R)$. By (2) and the definition of the Zariski topology, there exists $r \in R \setminus P$ such that $P \in X_r \subseteq \{Q \in \text{Spec}(R) \mid Q \subseteq P\}$. As $\{Q \in \text{Spec}(R) \mid Q \subseteq P\} \subseteq X_r$, (3) follows.

(3) \Rightarrow (1): Apply [14, Corollary 5.2] to the multiplicatively closed set generated by r . (For an alternate proof that (1) \Leftrightarrow (3), see [20, Proposition 1, p. 87].)

(3) \Rightarrow (4): Let $P \in \text{Spec}(R)$. The "quasilocal treed" hypothesis ensures that each prime ideal of R is comparable to P under inclusion. Thus, if P, r are as in (3), then $r \in \cap\{Q \in \text{Spec}(R) \mid P \subset Q\} \setminus P$.

(4) \Rightarrow (3): For $P \in \text{Spec}(R)$, use (4) to find $r \in \cap\{Q \in \text{Spec}(R) \mid P \subset Q\} \setminus P$. Then P, r are as in (3).

(4) \Leftrightarrow (5): Let $P \in \text{Spec}(R)$. If $P \in \text{Max}(R)$, the "empty intersection" $\cap\{Q \in \text{Spec}(R) \mid P \subset Q\}$ is interpreted as R , in which case P vacuously satisfies the conditions in both (4) and (5). Next, suppose that P is nonmaximal. Then

$W := \bigcap \{Q \in \text{Spec}(R) \mid P \subset Q\}$ is a prime ideal of R (since it is the intersection of a chain of prime ideals of R [17, Theorem 9]), and it is clear that no prime ideal of R can be properly contained between P and W . Accordingly, since $P \subseteq W$, it suffices to observe that $P \neq W$ if and only if P and W are adjacent in $\text{Spec}(R)$. \square

The conditions in Theorem 2.5 are not automatically satisfied. For instance, it is well known that there exist valuation domains R (which are certainly quasilocal treed rings such that $Z(R) = \text{Nil}(R)$ but) which do not satisfy condition (5) (cf. [18]). However, we close the section by recording the fact that those conditions are satisfied in the finite-dimensional case.

COROLLARY 2.6. *Let R be a quasilocal treed ring such that $Z(R) = \text{Nil}(R)$ and $\dim(R) < \infty$. Then for each $P \in \text{Spec}(R)$, there exists (a non-zero-divisor) $r \in R \setminus P$ such that $R_P = R_r$.*

Proof. It suffices to verify condition (5) in Theorem 2.5. If $P \in \text{Spec}(R) \setminus \text{Max}(R)$ has height n , let Q be the height $n + 1$ prime ideal of R , and observe that $P \subset Q$ are adjacent in $\text{Spec}(R)$. \square

3 LOCALLY PSEUDO-VALUATION RINGS

It is convenient to develop the first few results in this section for the quasilocal case, that is, in the context of *PVRs*. Recall that a ring R is said to be a *pseudo-valuation ring (PVR)* if, for each $P \in \text{Spec}(R)$ and $a, b \in R$, one has that Pa and Rb are comparable with respect to inclusion. Note that an integral domain is a *PVR* if and only if it is a pseudo-valuation domain (*PVD*). We often use the facts that the class of *PVRs* is stable under the formation of localizations (at prime ideals) [1, Theorem 12, Corollary 4] and homomorphic images [1, Corollary 3]. In particular, each *PVR* is an *LPVR* (and so the *PVRs* may be characterized as the quasilocal *LPVRs*.) While the focus in Theorem 3.1 emulates the interest in [3] on idealization, it also retains the “ $Z(A) = \text{Nil}(A)$ ” flavor of Section 2, since this equality holds for $A = R(+)E$ whenever R is an integral domain.

THEOREM 3.1. *Let R be a ring and E an R -module. Put $A := R(+)E$. Then:*

- (a) *If A is a PVR, then R is a PVR and E is a divisible R -module*
- (b) *If R is a PVD and E is a divisible R -module, then A is a PVR.*

Proof. (a) Since A is a *PVR*, so is its homomorphic image $A/(0(+)E) \cong R$. It remains to show that E is a divisible R -module. We show that if r is a regular element of R and $e \in E$, then there exists $f \in E$ such that $rf = e$. Without loss of generality, r is a nonunit of R , and so $r \in P$ for some $P \in \text{Spec}(R)$. Put $Q := P(+)E \in \text{Spec}(A)$, and consider the elements $a := (r, 0)$ and $b := (0, e)$ of A . Easy calculations show that $Qa = Pr(+)rE$ and $Ab = 0(+)Re$. If $Qa \subseteq Ab$, then $Pr = 0$, whence $P = 0$ (since r is regular) and so $r = 0$ by the choice of P , contradicting the condition that r is regular. Hence, since A is a *PVR*, we have that $Ab \subseteq Qa$. In particular, $Re \subseteq rE$, and so a suitable f can be found.

(b) Let $Q \in \text{Spec}(A)$. By [16, Theorem 25.1 (3)], $Q = P(+)E$ for some $P \in \text{Spec}(R)$. Suppose that elements $u = (b, e)$ and $v = (c, f)$ of A are such that $u \notin Qv$. Our task is to show that $Qv \subseteq Au$.

Consider first the case that that $b \notin Pc$. Since R is a *PVR*, we have $Pc \subseteq Rb$.

Thus if $p \in P$, there exists $r \in R$ such that $pc = rb$. It suffices to show that if $h \in E$, then there exists $g \in E$ such that $(p, h)v = (r, g)u$; that is, such that $pf + ch = re + bg$. As $b \neq 0$ and E is a divisible module over the integral domain R , a suitable g can be found.

In the remaining case, $b = pc$ for some $p \in P$. By hypothesis, there does not exist $h \in E$ such that $u = (p, h)v$; that is, such that $pf + ch = e$. Since E is R -divisible, $c = 0$, whence $b = 0$. As $u = (0, e) \notin Qv = 0(+)Pf$, it follows from divisibility of E that $P = 0$. Hence, $Qv = 0 \subseteq Au$, as desired. \square

It is important to remark that the converse of Theorem 3.1 (a) is false. To see this, use [2, Example 3.16 (c)] to construct a pseudo-valuation ring (R, M) such that $Nil(R) \subset Z(R) = M$. Take $E := R$ in the definition of A in Theorem 3.1. Pick $\beta \in Z(R) \setminus Nil(R)$; put $b := (\beta, 0)$, $a := (0, 1)$, and $P := Nil(R)(+)R = Nil(A) \subseteq A$. Since a zero-divisor cannot be a unit, an easy calculation shows that b does not divide a in A . If A were a PVR , then a characterization of PVR s [1, Theorem 5] would imply that a divides b^2 in A , a contradiction since $\beta^2 \neq 0$. Hence, A is not a PVR . However, the choice of R ensures that $E = R$ is a divisible R -module. In other words, Theorem 3.2 (a) would be false if its "PVD" hypothesis were replaced by "PVR".

We turn next to a class of idealizations $R(+)E$ in which the "PVR" property can be characterized without having to specify that R is an integral domain.

THEOREM 3.2. *Let R be a ring and E an overring of R (viewed as an R -module in the usual way). Then the following conditions are equivalent:*

- (1) $A := R(+)E$ is a PVR ;
- (2) R is a PVR , $E = tq(R)$, and if R is not an integral domain, then $Nil(R)$ is the only prime ideal of R , $Nil(R)^2 = 0$, and there exists $w \in Nil(R)$ such that $Nil(R) \setminus \{0\} = \{uw \in R \mid u \in R \setminus Nil(R)\}$.

Proof. (1) \Rightarrow (2): Assume (1). Then by Theorem 3.1 (a), R is a PVR and E is a divisible R -module. Therefore, $E = tq(R)$, since $tq(R)$ is the only R -divisible overring of R . Now, suppose that R is not an integral domain. Then, since R is a quasilocal treed ring, $Nil(R) \in Spec(R)$ (cf. [17, Theorem 9]), and so $Nil(R) \neq 0$. Fix a nonzero element $w \in Nil(R)$. If w has index of nilpotency n , there is no harm in replacing w with w^{n-1} , and so we can suppose, without loss of generality, that $w^2 = 0$.

We claim that there does not exist a nonunit $r \in R \setminus Nil(R)$. Deny. Since $w \neq 0$, it is easy to check that $(0, 1)$ does not divide $(w, 0)$ in A . As $(r, 0)$ is a nonunit of A and A is a PVR , it follows from [1, Theorem 5] that $(w, 0)$ divides $(0, 1)(r, 0) = (0, r)$ in A , whence $r \in Ew \subseteq Nil(E)$, a contradiction. This proves the above claim. As every element in $R \setminus Nil(R)$ is therefore a unit of R , it follows that R has a unique prime ideal, namely, $Nil(R)$. Hence $dim(R) = 0$, and so $R = tq(R) = E$.

Consider any nonzero element $d \in Nil(R)$. By reasoning as above, we see that $d \in Ew = Rw$, whence $Nil(R) = Rw$. Write $d = uw$, with $u \in R$. It remains only to observe that $u \notin Nil(R) = Rw$, and this follows since $d \notin 0 = (Rw)w$.

(2) \Rightarrow (1): Assume (2). By Theorem 3.1 (b), we may suppose that R is not an integral domain. If w is as in (2), observe that $w^2 = 0$. (Otherwise, one sees via induction that if n is any positive integer, then w^n is the product of w with a unit

of R , contradicting the fact that w is a nonzero nilpotent element.) Accordingly, there are but three types of elements of R , namely, 0, a unit u of R , and a product uw of a unit of R with w . As a result, there are but nine types of elements of $R(+R) = A$. Since $Rw = \text{Nil}(R)$, routine calculations reveal that as b ranges over the elements of A , the ideal Ab takes on one of the following six forms: 0 , $0(+)\text{Nil}(R)$, $0(+R)$, $\text{Nil}(R)(+)\text{Nil}(R)$, $\text{Nil}(R)(+R)$, and $R(+R)$. On the other hand, consider $P := \text{Nil}(R)(+R)$, the unique prime ideal of A . As a ranges over the elements of A , routine calculations reveal that Pa takes on one of the following three forms: 0 , $0(+)\text{Nil}(R)$, and $\text{Nil}(R)(+R)$. Since each of these three forms is comparable under inclusion to each of the nine forms that Ab can attain, it follows from the very definition of "PVR" that A is a PVR. \square

It may be of interest to record an explicit example of a non-domain R of the kind described in condition (2) of Theorem 3.2. One such ring is $R := (\mathbb{Z}/2\mathbb{Z})[X]/(X^2)$, the ring of dual numbers over the field with two elements; in this example, the coset represented by X is a suitable w .

We next use Theorem 3.2 to construct a new family of PVRs.

COROLLARY 3.3. *Let p be a positive prime number and let n be a positive integer. Then $\mathbb{Z}/p^n\mathbb{Z}(+)\mathbb{Z}/p^n\mathbb{Z}$ is a PVR if and only if n is either 1 or 2.*

Proof. It is straightforward to check that that $R := \mathbb{Z}/p^n\mathbb{Z}$ is a PVR with a unique prime ideal; and, being a finite ring, R coincides with its total quotient ring. Let A denote the idealization in question. Hence, by Theorem 3.2, A is a PVR if $n = 1$. Next, since $\text{Nil}(R) = p\mathbb{Z}/p^n\mathbb{Z}$, note that $\text{Nil}(R)^2 = 0$ if and only if $n \leq 2$. Finally, observe that if $n = 2$, then the coset represented by p is a suitable w , so that an application of Theorem 3.2 completes the proof. \square

The next result collects some facts about LPVRs that are analogous to the behavior of the "locally divided" concept, while also providing the promised examples of LPVRs that arise as idealizations.

PROPOSITION 3.4. (a) *Each LPVR is a locally divided ring.*

(b) *If $P \subseteq Q$ are prime ideals of a ring R and R_Q is a PVR, then R_P is a PVR.*

(c) *A ring R is an LPVR if and only if R_M is a PVR for each $M \in \text{Max}(R)$.*

(d) *The class of LPVRs is stable under the formation of rings of fractions and homomorphic images.*

(e) *Let R_1, \dots, R_n be finitely many rings, and put $A = R_1 \times \dots \times R_n$. Then A is an LPVR if and only if R_i is an LPVR for each $i = 1, \dots, n$.*

(f) *Let R be an integral domain and E an R -module. Then $R(+)E$ is an LPVR if and only if R is an LPVD and E_M is a divisible R_M -module for each $M \in \text{Max}(R)$.*

(g) *Let R be a ring such that $Z(R) = \text{Nil}(R)$ and E an overring of R . Then the following conditions are equivalent:*

(1) *$A := R(+)E$ is an LPVR;*

(2) *R is an LPVR, $E = \text{tq}(R)$, and if R is not an integral domain, then R is a PVR such that $\text{Nil}(R)$ is the only prime ideal of R , $\text{Nil}(R)^2 = 0$, and there exists $w \in \text{Nil}(R)$ such that $\text{Nil}(R) \setminus \{0\} = \{uw \in R \mid u \in R \setminus \text{Nil}(R)\}$.*

Proof. (a) The assertion is immediate from the definitions, since each PVR is a

divided ring.

(b) It suffices to observe that $R_P \cong (R_Q)_{PR_Q}$ [5, Proposition 7 (i), p. 65] is a localization of a *PVR*.

(c) The "only if" assertion is trivial, and the "if" assertion follows from (b).

(d), (e): Using the above remarks, one easily adapts the proof of the corresponding facts about the class of locally divided rings [3, Proposition 2.1 (a), (b)].

(f) Let $A : +R(+)E$. A proof is easily fashioned by combining the following four facts: $Max(A) = \{M(+)E \mid M \in Max(R)\}$ [16, Theorem 25.1]; if $M \in Max(R)$ and $N := M(+)E$, then $A_N = R_M(+)E_M$ [16, Corollary 25.5 (2)]; part (c) of the present result; and Theorem 3.1.

(g) Let P_0 denote the unique minimal prime ideal of R . As recalled in the Introduction, $T := R_{P_0} = tq(R)$. If $M \in Max(R)$, then [3, Proposition 2.5 (c)] and [5, Proposition 7 (i), p. 65] provide canonical inclusions $R \subseteq R_M \subseteq E_M := E_{R \setminus M} \subseteq T_{R \setminus M} = (R_{P_0})_{R \setminus M} = R_{P_0} = T$. Thus, $T = tq(R_M)$ and E_M is an overring of R_M (and of R) for each $M \in Max(R)$.

(1) \Rightarrow (2): Assume (1). If $M \in Max(R)$ and $N := M(+)E$, then $A_N = R_M(+)E_M$ is a *PVR*. Therefore, by Theorem 3.2, R_M is a *PVR* and $E_M = tq(R_M) = T = T_M$. It follows that R is an *LPVR* by (c); and that $E = T$ by globalization. Suppose now that R is not an integral domain. In view of Theorem 3.2, it suffices to show that R is quasilocal. For each $M \in Max(R)$, observe that R_M is not a domain (since it has R as a subring), and so Theorem 3.2 ensures that $dim(R_M) = 0$. Hence, $dim(R) = 0$. As R has a unique minimal prime ideal, it follows that R has a unique prime ideal and, *a fortiori*, R is quasilocal, as desired.

(2) \Rightarrow (1): Assume (2). Since any *PVR* is an *LPVR*, Theorem 3.2 reduces us to the case in which R is an integral domain. For this case, argue as in the proof of (f) (letting either Theorem 3.1 or Theorem 3.2 play the role of Theorem 3.1 in the earlier proof). \square

Recall that any zero-dimensional ring is locally divided [3, Corollary 2.2]. Example 3.5 (a) shows that the conclusion cannot be strengthened to "*LPVR*". Moreover, Example 3.5 (b) provides an example of a nonquasilocal zero-dimensional *LPVR*.

EXAMPLE 3.5. (a) There exists a ring R such that $dim(R) = 0$ (so that R is a locally divided ring) and R is not an *LPVR*.

(b) There exists a ring R such that $dim(R) = 0$, R is an *LPVR*, and R is not a *PVR*.

Proof. (a) Let R be any idealization of the kind considered in Corollary 3.3, with $n \geq 3$. By Corollary 3.3, R is not a *PVR*. However, by [16, Theorem 25.1 (3)], R is quasilocal, and so R is not an *LPVR*. Of course, like any finite ring, R is zero-dimensional.

(b) Let R be the direct product of finitely many, but at least two, rings of the form $\mathbb{Z}/n\mathbb{Z}$, $n \geq 2$. Observe that R , being finite, is necessarily zero-dimensional; and, by Proposition 3.4 (e), that R is an *LPVR*. However, since $n \geq 2$, we see that R is not quasilocal and, *a fortiori*, not a *PVR*. \square

Proposition 3.6 is an *LPVR*-theoretic analogue of the result [3, Corollary 2.24 (e)] that if a ring R is such that the large quotient ring $R_{[P]}$ is a divided ring for

each $P \in \text{Spec}(R)$, then R is a locally divided ring.

PROPOSITION 3.6. *Let R be a ring such that $R_{[P]}$ is a PVR for each $P \in \text{Spec}(R)$. Then R is an LPVR.*

Proof. We show that R_Q is a PVR for each $Q \in \text{Spec}(R)$. Suppose first that Q is a regular prime ideal of R . By hypothesis, $R_{[P]}$ is a divided ring for each $P \in \text{Spec}(R)$. Then, by [3, Theorem 2.23 (c)], $Z(R) \subseteq Q$, and so by [3, Proposition 2.5 (a)], $R_Q = R_{[Q]}$, which was assumed to be a PVR.

In the remaining case, $Q \subseteq Z(R)$. Observe that $T := tq(R)$ is a ring of quotients of any $R_{[P]}$ and so, by Proposition 3.4 (d), T is an LPVR. Hence, so is $T_{QT} \cong R_Q$ [5, Proposition 7 (i), p. 65]; that is, R_Q is a quasilocal LPVR, namely, a PVR, to complete the proof. \square

We close with an example showing that the converse of Proposition 3.6 is false. Consider the ring $R := \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. As $\mathbb{Z}/4\mathbb{Z}$ is a PVR, hence an LPVR, it follows from Proposition 3.4 (e) that R is an LPVR. However, $R_{[P]}$ is not a PVR for any $P \in \text{Spec}(R)$. Indeed, the zero-dimensionality of R forces $tq(R) = R$, whence $R_{[P]} = R$, which is not quasilocal and, thus, not a PVR.

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